Math 245B Lecture 5 Notes

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1 Tietze's Extension Theorem and Compactness

1.1 Tietze's extension theorem

Let X be a normal topological space.

Theorem 1.1 (Tietze's extension theorem)**.** *Let $(X, \mathcal{T})$ be T_4, let $A \subseteq X$ be closed, and let $f \in C(A, [a, b])$. Then there exists $F \in C(X, [a, b])$ such that $F|_A = f$. The same holds if $C(X, [a, b])$ is replaced with $C(X, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$.*

Proof. Without loss of generality, translate so that $a = 0$. We claim that if $f \in C(A, [0, b])$, then there exists $g \in C(X, [0, b/3])$ such that $0 \leq f - g \leq 2b/3$. Let $B = \{x \in A : f(x) \leq b/3\}$, and let $C = \{x \in A : f(x) \geq 2b/3\}$. These are relatively closed in A, and since A is closed, they are closed in X. By Urysohn's lemma, there exists $g \in C(X, [0, b/3])$ such that $g|_B = 0$ and $g|_C = b/3$. Now check that

1. $g|_A \leq f$,
2. $f \leq g|_A + 2b/3$.

Let g_1 be given by the claim, and let $f_1 = f - g_1|A$. Apply the claim again. There exists $g_2 \in C(X, [0, 2/3 \cdot b/3])$ such that $0 \leq f_1 - g_2|_A \leq (2/3)^2 b$. By recursion, we find $g_n \in C(X, [0, (2/3)^{n-1} \cdot b/3])$, and $f - (\sum_{i=1}^n g_i)|_A \leq (2/3)^n b$. Now, for any $m \geq n \geq n$,

$$\left\| \sum_{i=1}^m g_i - \sum_{i=1}^m g_i \right\|_u = \left\| \sum_{i=n+1}^m g_i \right\|_u \leq \sum_{n+1}^m \|g_i\|_u \leq \sum_{i=n+1}^m (2/3)^{i-1} \frac{b}{3} \leq C(2/3)^n b.$$

So $F := \sum_{i=1}^\infty g_i \in C(X, [0, b])$, and if $x \in A$,

$$|f(x) - F(x)| = \lim_{n \to \infty} |f(x) - \sum_{i=1}^n g_i(x)| = 0.$$

Now suppose $f \in C(X, \mathbb{R})$. Consider $f' = f/(1 + |f|) \in C(X, (-1, 1))$. This has an extension $F' \in C(X, [-1, 1])$. Let $H = \{x : F'(x) = \pm 1\}$. This is closed and disjoint from

A. So by Urysohn's lemma, there exists $h \in C(X, [0,1])$ such that $h|_A = 1$ and $h|_H = 0$. Let $G = F' \cdot h$. Now $G \in C(X, (-1,1))$, and $G|_A = f'$. Now define $F := G/(1-|G|)$. Then $F \in C(X, \mathbb{R})$ such that $F|_A = f$.

For $X = \mathbb{C}$, split into the real and imaginary parts of f. $\square$

1.2 Compact spaces

Definition 1.1. A topological space X is **compact** if every open cover has a finite subcover. The same is true for a subset of X. A subset $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Remark 1.1. The characterization of compactness in metric spaces using sequences turns out to be not as useful in analysis, even though it can be defined in point set topology in general.

Definition 1.2. We say a family $\mathcal{F} \subseteq \mathscr{P}(X)$ has the **finite intersection property (FIP)** if $F_1 \cap \cdots \cap F_m \neq \varnothing$ whenever $m \in \mathbb{N}$ and $F_1, \dots, F_m \in \mathcal{F}$.

Lemma 1.1. *A topological space X his compact if and only if every FIP family of closed sets $\mathcal{F}$ has $\bigcap \mathcal{F} \neq \varnothing$.*

Proof. ($\Longrightarrow$): Let $\mathcal{F}$ be an FIP family of closed sets. Let $\mathcal{U} = \{X \setminus F : F \in \mathcal{F}\}$. For any $X \setminus F_1, \dots, X \setminus F_m \in U$, we know that there exists $x \in F_1 \cap \cdots \cap F_m$, so $x \notin \bigcup_{i=1}^m (X \setminus F_i)$. So by compactness $\bigcup \mathcal{U} \neq X$. So $\bigcap \mathcal{F} \neq \varnothing$.

($\Longleftarrow$): The reverse implication is just the same steps, but in reverse order. $\square$

Proposition 1.1. *If X is compact and $A \subseteq X$ is closed, then A is compact.*

Proof. Suppose $\mathcal{U}$ is a family of open sets in X such that $A \subseteq \bigcup \mathcal{U}$. Define $\mathcal{V} = \mathcal{U} \cup \{X \setminus A\}$. This is an open cover of X, so it has a finite subcover $U_1 \dots, U_m \in \mathcal{U}$ such that $X = (X \setminus A) \cup \bigcup_{1=1}^m U_i$. So $U_1, \dots, U_m$ form a finite subcover of A. $\square$

1.3 Compact Hausdorff spaces

Some topologies, like the trivial topology, give us undesirable compact spaces. We add the condition of Hausdorff to get spaces we do want.

Proposition 1.2. *Let X be Hausdorff, let $F \subseteq X$ be compact, and let $x \in X \setminus F$. Then there exist disjoint neighborhoods $U \ni x$ and $V \subseteq F$.*

Proof. For all $y\ inF$, we have $y \neq x$, so there exist disjoint open sets $U_y \ni x$ and $V_y \ni y$. Now $F \subseteq \bigcup_y V_y$, so there exist $y_1, \dots, y_m \in F$ such that $F \subseteq V_{y_1} \cup \cdots V_{y_m}$. Now $F \subseteq V :$ $V_{y_1} \cup \cdots V_{y_m}$ is disjoint from $U := U_{y_1} \cap \cdots \cap U_{y_m}$. U is an open neighborhood of x. $\square$

Proposition 1.3. *A compact subset of a Hausdorff space is closed.*

Proof. If F is a compact subset of X, then every $x \in X\ setminusF$ admits an open $U \ni x$ such that $F \cap U = \varnothing$. So $X \setminus F = \bigcup u_x$, so F is closed. $\square$

Proposition 1.4. *A compact Hausdorff space is normal.*

Proof. Let $A, B \subseteq X$ be disjoint and closed. A and B are compact. So for all $x \in A$, there exist disjoint neighborhoods $U_x \ni x$ and $V_x \supseteq B$. Now $\{U_x : x \in A\}$ is an open cover. so there exist $U_{x_1} \cup \cdots, \cup U_{x_n} \supseteq A$ disjoint from $V_{x_1} \cap \cdots \cap V_{x_n} \supseteq B$. $\square$

1.4 Continuous functions on compact spaces

Proposition 1.5. *If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $f(X)$. Then $f^{-1}[\mathcal{U}] = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X. By compactness, there exists a finite subcover $f^{-1}[U_1], \ldots, f^{-1}[U_m]$. Then $U_1 \cup \cdots \cup U_m \supseteq f(X)$ is an open cover. $\square$

Corollary 1.1. *If $Y = \mathbb{R}$, then extreme values are obtained. So $C(X) = BC(X)$.*

Proposition 1.6. *Let X be compact, Y be Hausdorff, and let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism; i.e. f^{-1} is also continuous.*

Proof. Let $C \subseteq X$ be closed. Then C is compact, so $f(C)$ is compact. Then $f(C)$ is closed, as Y is Hausdorff. So f sends closed sets to closed sets; i.e. f^{-1} is continuous. $\square$